Fluctuations from Normalized Intensive Cumulants

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Introduction

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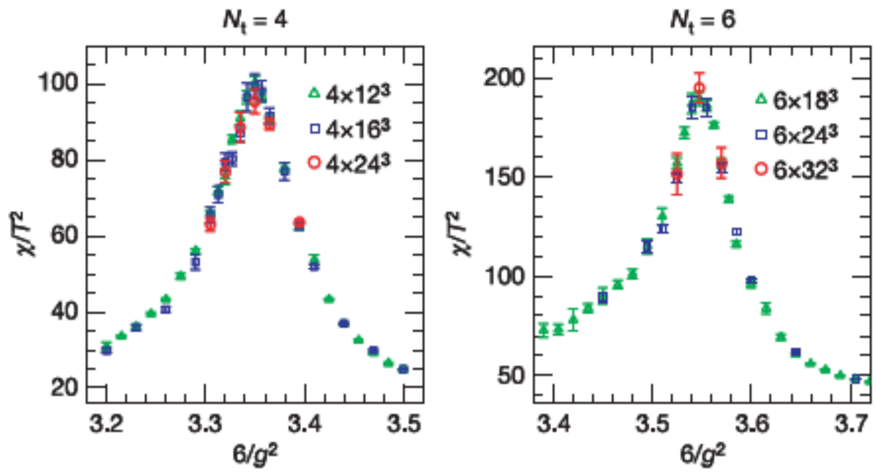

Figure 1 | Susceptibilities for the light quarks for $N_t = 4$ and for $N_t = 6$ as a function of $6/g^2$, where g is the gauge coupling. (T grows with $6/g^2$.) The largest volume is eight times bigger than the smallest one, so a first-order phase transition would predict a susceptibility peak that is eight times higher (for a second-order phase transition the increase would be somewhat less, but still dramatic). Instead of such a significant change we do not observe any volume dependence. Error bars are s.e.m.

susceptibilities (χ) and correlation lengths (ξ) diverge at the critical point.......

multiplicity moments and ξ.......

$\langle(\delta N)^2\rangle \sim \xi^2$

$\langle(\delta N)^3\rangle \sim \xi^{4.5}$

$\langle(\delta N)^4\rangle - 3\langle(\delta N)^2\rangle^2 \sim \xi^7$

Kurtosis $\times$ Variance $\sim \chi^{(4)}/ [\chi^{(2)}T^2]$

Skewness $\times$ Sigma $\sim [\chi^{(3)}T]/ [\chi^{(2)}T^2]$

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Using Higher Moments of Fluctuations and their Ratios in the Search for the QCD Critical Point

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The QCD critical point can be found in heavy ion collision experiments via the non-monotonic behavior of many fluctuation observables as a function of the collision energy. The event-by-event fluctuations of various particle multiplicities are enhanced in those collisions that freeze out near the critical point. Higher, non-Gaussian, moments of the event-by-event distributions of such observables are particularly sensitive to critical fluctuations, since their magnitude depends on the critical correlation length to a high power. We present quantitative estimates of the contribution of critical fluctuations to the third and fourth moments of the pion, proton and net proton multiplicities, as well as estimates of various measures of pion-proton correlations, all as a function of the same five non-universal parameters, one of which is the correlation length that parametrizes proximity to the critical point. We show how to use nontrivial but parameter independent ratios among these more than a dozen fluctuation observables to discover the critical point. We also construct ratios that, if the critical point is found, can be used to overconstrain the values of the non-universal parameters.

Variables

Two passes through the data

One to get <x>, the next for δx…

Deviations:

$$\delta x \equiv x - \langle x\rangle$$
$$\delta y \equiv y - \langle y\rangle$$

Cumulants:

We now define the cumulants of the event-by-event distribution of a single observable, say x. The second and third cumulants are given by

$$\kappa_{2x} \equiv \langle\langle x^2\rangle\rangle \equiv \langle\, (\delta x)^2\, \rangle \tag{1.2}$$
$$\kappa_{3x} \equiv \langle\langle x^3\rangle\rangle \equiv \langle\, (\delta x)^3\, \rangle\ , \tag{1.3}$$

where we have introduced two equivalent notations for the cumulants. The second cumulant κ_{2x} is the variance of the distribution, while the skewness of the distribution is given by $\kappa_{3x}/\kappa_{2x}^{3/2}$. The fourth cumulant is different from the corresponding fourth moment:

$$\kappa_{4x} \equiv \langle\langle x^4\rangle\rangle \equiv \langle\, (\delta x)^4\, \rangle - 3\,\langle\, (\delta x)^2\, \rangle^2\ . \tag{1.4}$$

Mixed Cumulants:

$$\kappa_{1x1y} \equiv \langle\langle xy\rangle\rangle = \langle\, \delta x\, \delta y\, \rangle\ , \tag{1.8}$$
$$\kappa_{1x2y} \equiv \langle\langle xy^2\rangle\rangle = \langle\, \delta x\, (\delta y)^2\, \rangle\ , \tag{1.9}$$
$$\kappa_{2x2y} \equiv \langle\langle x^2y^2\rangle\rangle = \langle\, (\delta x)^2\, (\delta y)^2\, \rangle - 2\langle\, \delta x\, \delta y\, \rangle^2 - \langle\, (\delta x)^2\, \rangle\,\langle\, (\delta y)^2\, \rangle\ , \tag{1.10}$$
$$\kappa_{1x3y} \equiv \langle\langle xy^3\rangle\rangle = \langle\, \delta x\, (\delta y)^3\, \rangle - 3\,\langle\, \delta x\, \delta y\, \rangle\,\langle\, (\delta y)^2\, \rangle\ . \tag{1.11}$$

Intensive Normalized Cumulants:

$$\omega_{i\pi} \equiv \frac{\kappa_{i\pi}}{\langle N_\pi\rangle}\ , \tag{1.12}$$
$$\omega_{ip} \equiv \frac{\kappa_{ip}}{\langle N_p\rangle}\ , \tag{1.13}$$
$$\omega_{i(p-\bar{p})} \equiv \frac{\kappa_{i(p-\bar{p})}}{\langle N_p + N_{\bar{p}}\rangle}\ , \tag{1.14}$$
$$\omega_{ipj\pi} \equiv \frac{\kappa_{ipj\pi}}{\langle N_p\rangle^{i/r}\langle N_\pi\rangle^{j/r}}\ , \tag{1.15}$$
$$\omega_{i(p-\bar{p})j\pi} \equiv \frac{\kappa_{i(p-\bar{p})j\pi}}{\langle N_p + N_{\bar{p}}\rangle^{i/r}\langle N_\pi\rangle^{j/r}}\ , \tag{1.16}$$

where $r \equiv i + j$.

Variables

$$\kappa_{1x1y} \equiv \langle\langle xy \rangle\rangle = \langle\, \delta x\, \delta y \,\rangle \;, \tag{1.8}$$

$$\kappa_{1x2y} \equiv \langle\langle xy^2 \rangle\rangle = \langle\, \delta x\, (\delta y)^2 \,\rangle \;, \tag{1.9}$$

$$\begin{aligned} \kappa_{2x2y} &\equiv \langle\langle x^2 y^2 \rangle\rangle \\ &= \langle\, (\delta x)^2\, (\delta y)^2 \,\rangle - 2\langle\, \delta x\, \delta y \,\rangle^2 - \langle\, (\delta x)^2 \,\rangle \langle\, (\delta y)^2 \,\rangle \;, \end{aligned} \tag{1.10}$$

$$\begin{aligned} \kappa_{1x3y} &\equiv \langle\langle xy^3 \rangle\rangle \\ &= \langle\, \delta x\, (\delta y)^3 \,\rangle - 3\, \langle\, \delta x\, \delta y \,\rangle\, \langle\, (\delta y)^2 \,\rangle \;. \end{aligned} \tag{1.11}$$

x (or y) is π, p, or p-pbar

$$\omega_{i\pi} \equiv \frac{\kappa_{i\pi}}{\langle N_\pi \rangle} \;, \tag{1.12}$$

$$\omega_{ip} \equiv \frac{\kappa_{ip}}{\langle N_p \rangle} \;, \tag{1.13}$$

$$\omega_{i(p-\bar{p})} \equiv \frac{\kappa_{i(p-\bar{p})}}{\langle N_p + N_{\bar{p}} \rangle} \;, \tag{1.14}$$

$$\omega_{ipj\pi} \equiv \frac{\kappa_{ipj\pi}}{\langle N_p \rangle^{i/r} \langle N_\pi \rangle^{j/r}} \;, \tag{1.15}$$

$$\omega_{i(p-\bar{p})j\pi} \equiv \frac{\kappa_{i(p-\bar{p})j\pi}}{\langle N_p + N_{\bar{p}} \rangle^{i/r} \langle N_\pi \rangle^{j/r}} \;, \tag{1.16}$$

where $r \equiv i + j$.

If the N_x's are statistically independent

&	Gaussian	Poisson
	ω_2's $\neq 0$	ω_i $(i \geq 2) = 1$
	others = 0	others = 0

Dependence of ξ on μ_B

The simplest ansatz for $\xi(\mu_B)$ that we have found that incorporates the physics that we have just described is

$$\xi(\mu_B) = \frac{\xi_{\max}}{\left[1 + \frac{(\mu_B - \mu_B^c)^2}{W(\mu_B)^2}\right]^{1/3}} , \qquad (1.17)$$

with

$$W(\mu_B) = W + \delta W \tanh\left(\frac{\mu_B - \mu_B^c}{w}\right) \qquad (1.18)$$

where W and w are nonuniversal parameters to be chosen and δW is specified by requiring that

$$\frac{W + \delta W}{W - \delta W} = \left(\frac{f_+}{f_-}\right)^{3/2} = 1.9^{3/2} . \qquad (1.19)$$

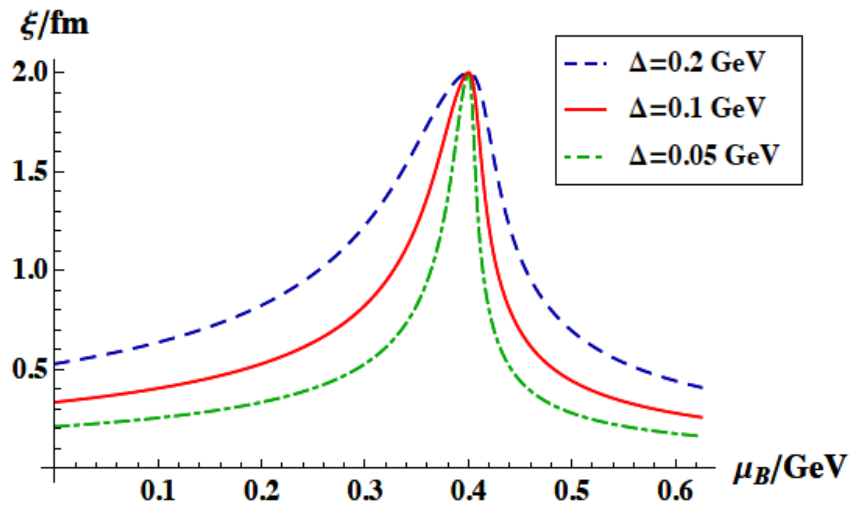
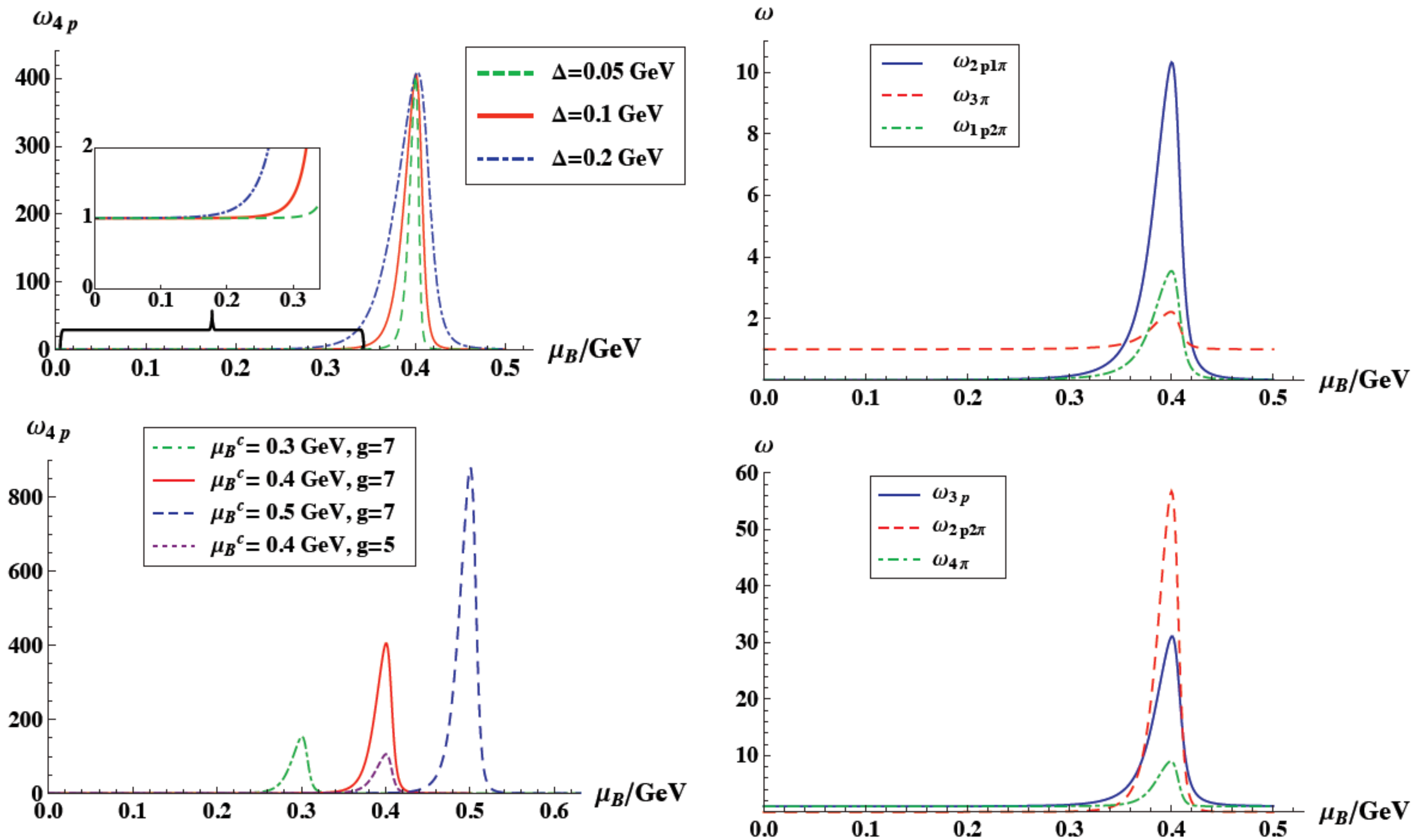


FIG. 1: The correlation length $\xi(\mu_B)$ achieved in a heavy ion collision that freezes out with a chemical potential μ_B, according to the ansatz described in the text. We have assumed that the collisions that freeze out closest to the critical point are those that freeze out at $\mu_B^c = 400$ MeV. We have assumed that the finite duration of the collision limits ξ to $\xi < \xi_{\max} = 2$ fm. We show $\xi(\mu_B)$ for three choices of the width parameter Δ, defined in the text. The choices of parameters that have gone into this ansatz are arbitrary, made for illustrative purposes only. They are not predictions.

fluctuations of σ field:	2.1-2.4
π & p coupling strengths:	2.5
correlators due to σ exchange	2.7-2.13
correlators to cumulants	II.B (2.19 & 2.25)
cumulant ratios	III

Dependence of ξ on μ_B



Heights of the different peaks are affected differently by the different "universal parameters":

V	volume	(constrained by centrality cuts)	
ξ	correlation length	(what we're trying to measure)	
g_p	sigma-proton coupling	benchmark value	= 7
G	sigma-pion coupling		= 300 MeV
λ_3	dimensionless σ^3 coupling		= 4
λ_4	dimensionless σ^4 coupling		= 12

Ratios of Cumulants

Given an observation of critical behavior, the values of the different cumulants can be used to overconstrain the values of the universal parameters…..

The contributions of critical fluctuations to different correlators depend on different combinations of ξ and the four parameters. For example,

$$\begin{aligned} \kappa_{2p,\sigma} &\sim V n_p^2\, g_p^2\, \xi^2, \\ \kappa_{3p,\sigma} &\sim V n_p^3\, g_p^3\, \tilde{\lambda}_3\, \xi^{9/2}, \\ \kappa_{4p,\sigma} &\sim V n_p^4\, g_p^4\, \tilde{\lambda}_4'\, \xi^7, \end{aligned} \tag{3.1}$$

where $\tilde{\lambda}_4' \equiv 2\tilde{\lambda}_3^2 - \tilde{\lambda}_4$. For the most general pion-proton cumulant,

$$\kappa_{ipj\pi,\sigma} \sim V n_p^i\, g_p^i\, g_\pi^j\, \tilde{\lambda}_r'\, \xi^{\frac{5}{2}r-3}\ , \tag{3.2}$$

with $r = i + j$ and with $\tilde{\lambda}_r'$ as defined in (2.22). We have kept the n_p-dependence since it introduces significant μ_B-dependence, but we have suppressed the T- and n_π-dependence. In Table I we present the parameter dependence of various cumulant ratios. Except for the first 3 entries, N_π, N_p and $\kappa_{ipj\pi}$, the quantities we consider are all V-independent (i.e. intensive) by construction. (In constructing intensive ratios, we can always remove V-dependence by dividing by N_π to the appropriate power.) Note that although we have not written the σ subscripts in the table, the table only describes the parameter-dependence of the contributions from critical fluctuations. When the ratios in the table are constructed from data, the Poisson contribution must be subtracted from each measured κ separately, before taking a ratio.

TABLE I: Parameter dependence of the contribution of critical fluctuations to various particle multiplicity cumulant ratios. We have subtracted the Poisson contribution from each cumulant before taking the ratio. The table shows the power at which the parameters enter in each case. We only considered cases with $r \equiv i+j = 2,\ 3,\ 4$. We defined $2\tilde{\lambda}_3^2 - \tilde{\lambda}_4 \equiv \tilde{\lambda}_4'$.

ratio	V	$n_p(\mu_B)$	g_p	g_π	$\tilde{\lambda}_3$	$\tilde{\lambda}_4'$	ξ
N_π	1	-	-	-	-	-	-
N_p	1	1	-	-	-	-	-
$\kappa_{ipj\pi}$	1	i	i	j	$\delta_{r,3}$	$\delta_{r,4}$	$\frac{5}{2}r-3$
$\omega_{ipj\pi}$	-	$i-\frac{i}{r}$	i	j	$\delta_{r,3}$	$\delta_{r,4}$	$\frac{5}{2}r-3$
$\kappa_{ipj\pi}N_\pi^{i-1}/N_p^i$	-	-	i	j	$\delta_{r,3}$	$\delta_{r,4}$	$\frac{5}{2}r-3$
$\kappa_{2p2\pi}N_\pi/\kappa_{4\pi}\kappa_{2p}$	-	-	-	-2	-	-	-2
$\kappa_{4p}N_\pi^2/\kappa_{4\pi}\kappa_{2p}^2$	-	-	-	-4	-	-	-4
$\kappa_{2p2\pi}N_p^2/\kappa_{4p}N_\pi^2$	-	-	-2	2	-	-	-
$\kappa_{3p1\pi}N_p/\kappa_{4p}N_\pi$	-	-	-1	1	-	-	-
$\kappa_{3p}N_p^{3/2}/\kappa_{2p}^{9/4}N_\pi^{1/4}$	-	-	$-3/2$	-	1	-	-
$\kappa_{2p}\kappa_{4p}/\kappa_{3p}^2$	-	-	-	-	-2	1	-
$\kappa_{3p}\kappa_{2\pi}^{3/2}/\kappa_{3\pi}\kappa_{2p}^{3/2}$	-	-	-	-	-	-	-
$\kappa_{4p}\kappa_{2\pi}^2/\kappa_{4\pi}\kappa_{2p}^2$	-	-	-	-	-	-	-
$\kappa_{4p}^3\kappa_{3\pi}^4/\kappa_{4\pi}^3\kappa_{3p}^4$	-	-	-	-	-	-	-
$\kappa_{2p2\pi}^2/\kappa_{4\pi}\kappa_{4p}$	-	-	-	-	-	-	-
$\kappa_{2p1\pi}^3/\kappa_{3p}^2\kappa_{3\pi}$	-	-	-	-	-	-	-

Analysis Introduction

Selected highlights from the paper....

ω_{ix} and ω_{ixjy} instead of $S\sigma$ & $K\sigma^2$, where x & y are π, p, & [p−pbar], and i & j = 2,3,4 ($r = i+j \leq 4$)...

ω_{ix} and ω_{ixjy} have well-defined values for Gaussian or Poisson statistics in the absence of a C.P.

some of these ω observables are expected to be more sensitive than others....

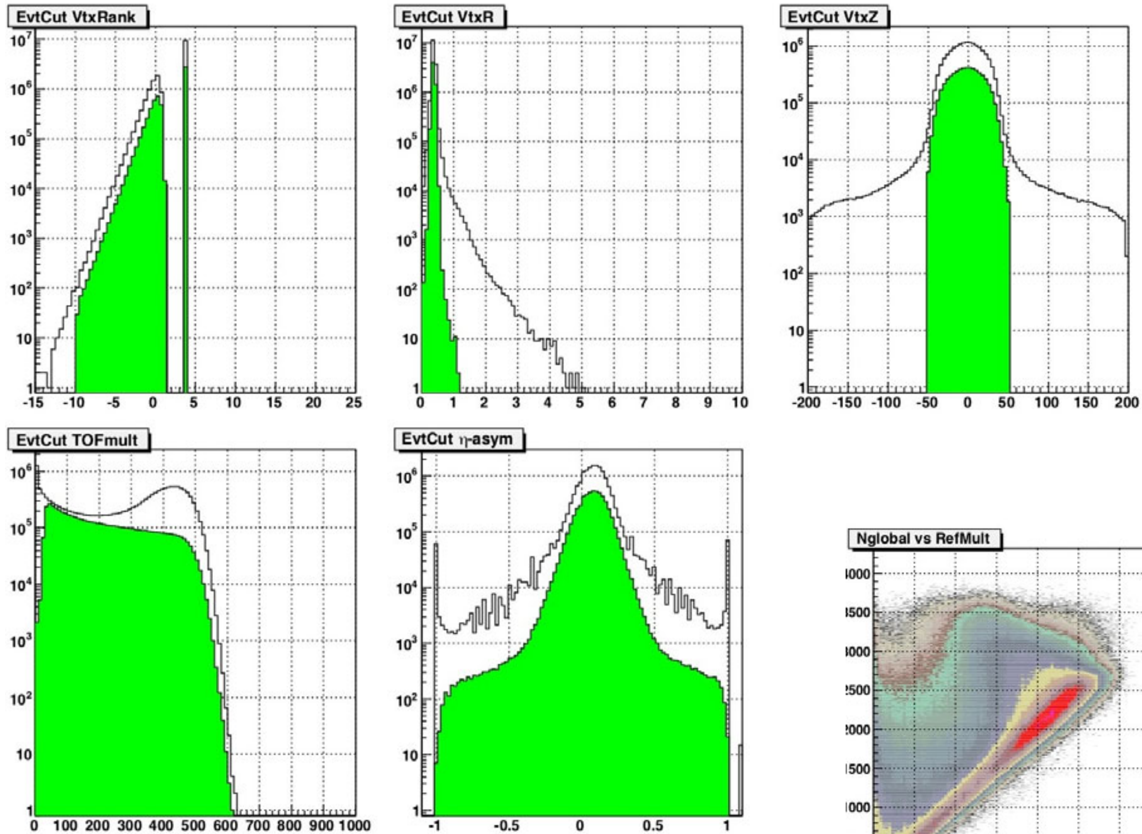
- $\omega_{ipj\pi}$ is more sensitive than $\omega_{i(p\text{-}pbar)j\pi}$ for any $i \neq 0$ and any j
- critical contribution to $\omega_{ipj\pi}$ is largest when r is largest (strongest ξ dependence)
- $\omega_{4p} >> \omega_{4\pi}$ and ω_{4p} is the most sensitive of all cumulants using the "benchmark" model parameters
- if $(\omega_{4p} - 1) < 1$ at μ_B~400 MeV, then ξ<1 fm at that value of μ_B
- if μ_B^c < 400 MeV and/or g_p/g_π is smaller than the benchmark values, then $\omega_{4\pi}$ is the most sensitive...

ω_{ix} and ω_{ixjy} diverge at the critical point by different amounts depending on various σ model parameters

i.e. the observed values can be used to mathematically overconstrain these parameters...

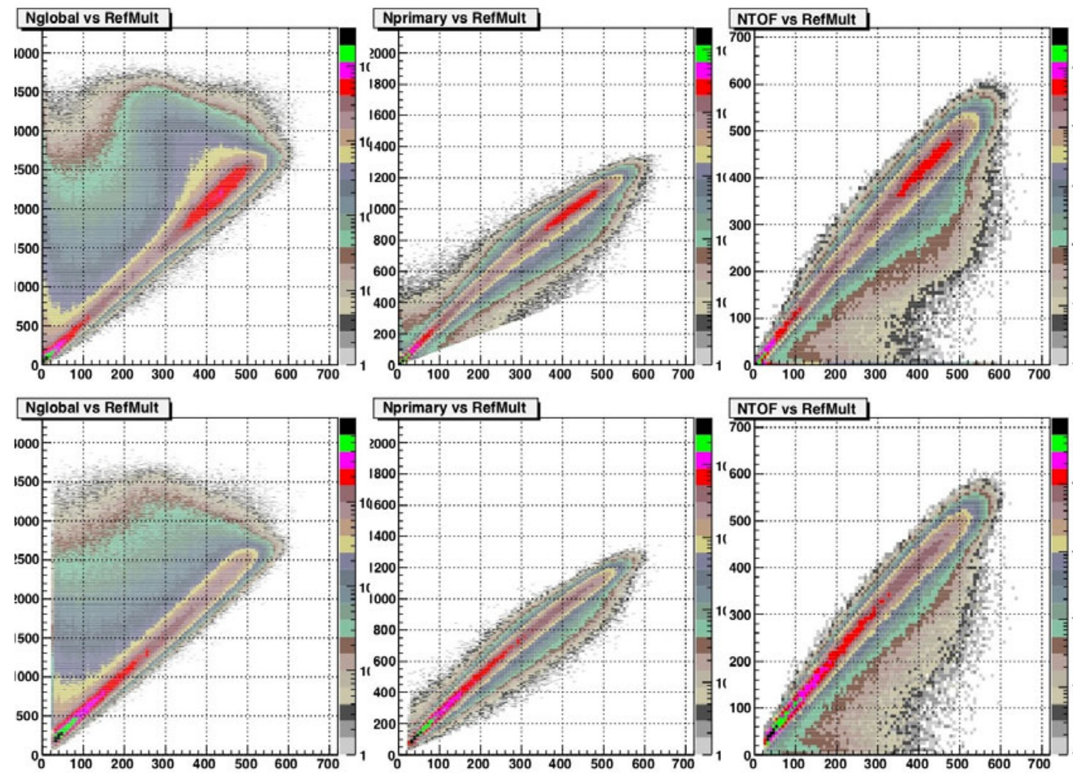
- Daniel and I have two independent codes to analyse the BES data in these directions…
- Some results from my code are now presented for the 7.7, 11, 39, 200 GeV data…
- This is just a basic investigation so far… …lots to do still
  - I am just using "dE/dx – TOF" PID (details later)
  - Daniel is using both "dE/dx – TOF" & "dE/dx + TOF"…

Event Cuts

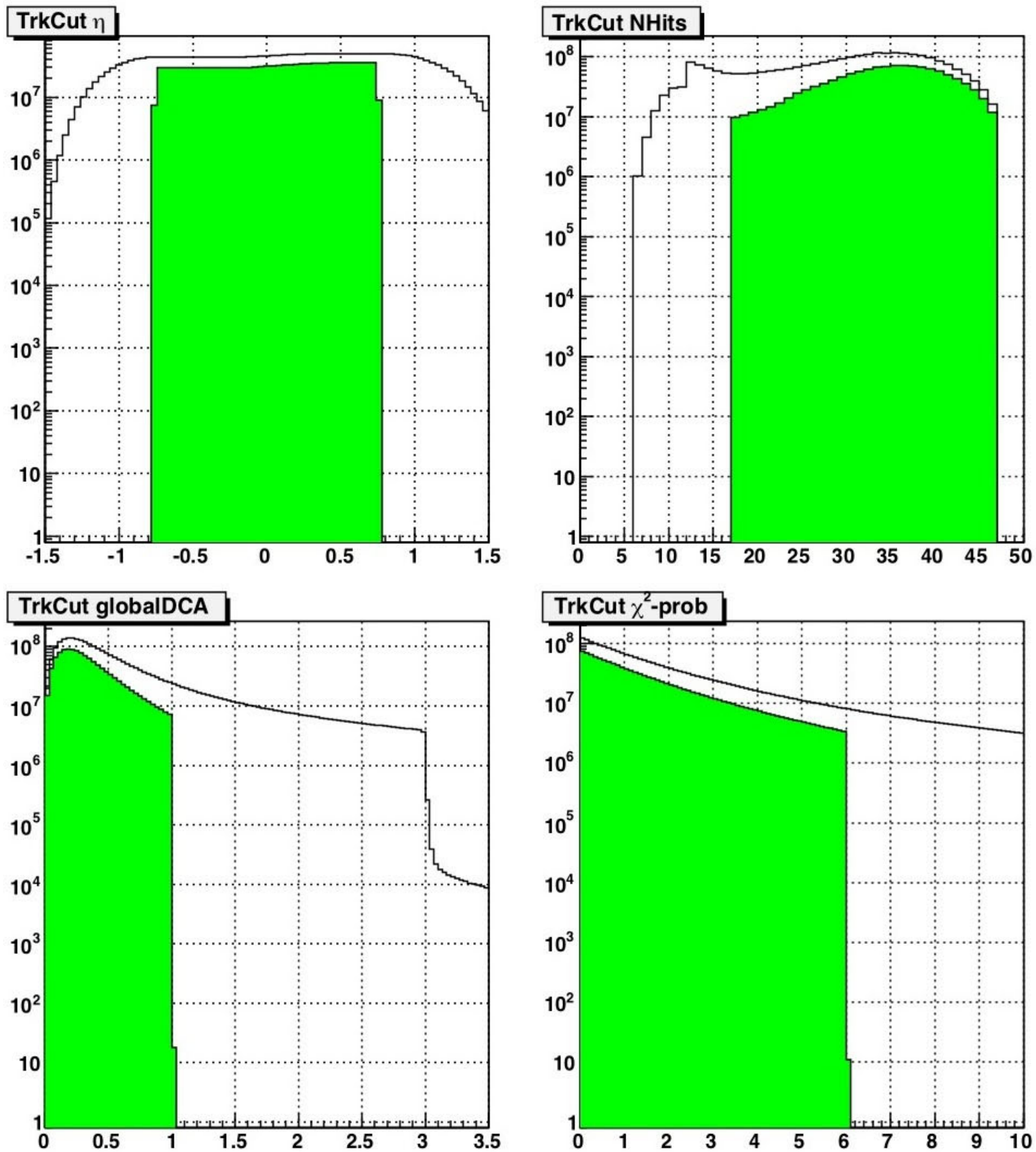


200 GeV data shown….

Rvtx ≤ 1.2cm
Zvtx ≤ 50cm
Tofmult ≥ 1



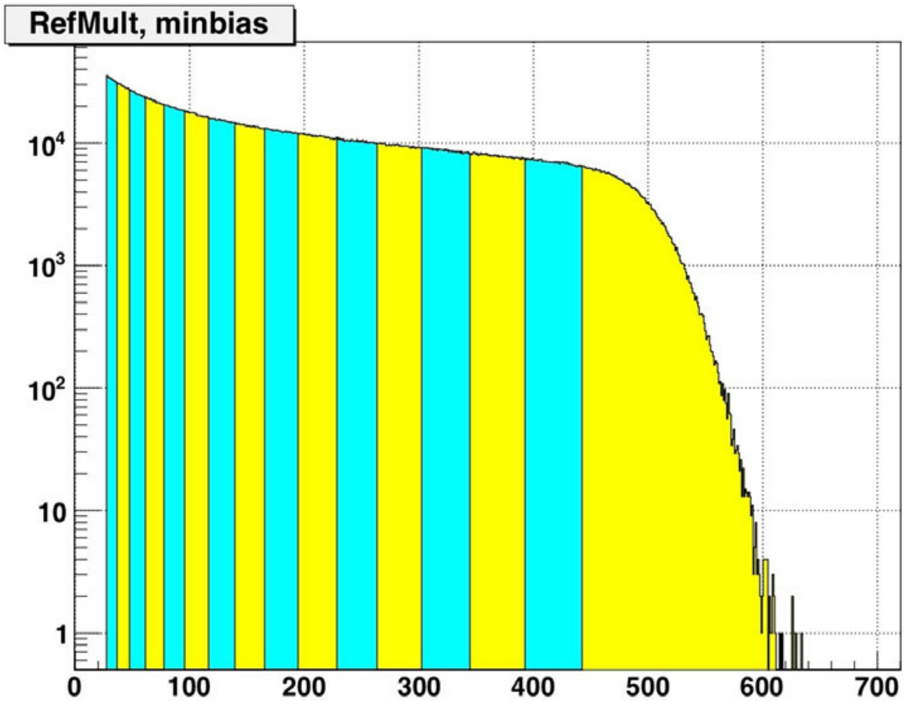
Track Cuts



200 GeV data shown…

$|\eta| \leq 0.75$
Nhitsfit ≥ 17
Nhitsposs ratio ≥ 0.52
DCAglobal < 1cm
χ^2 prob < 6

Centrality & Triggers



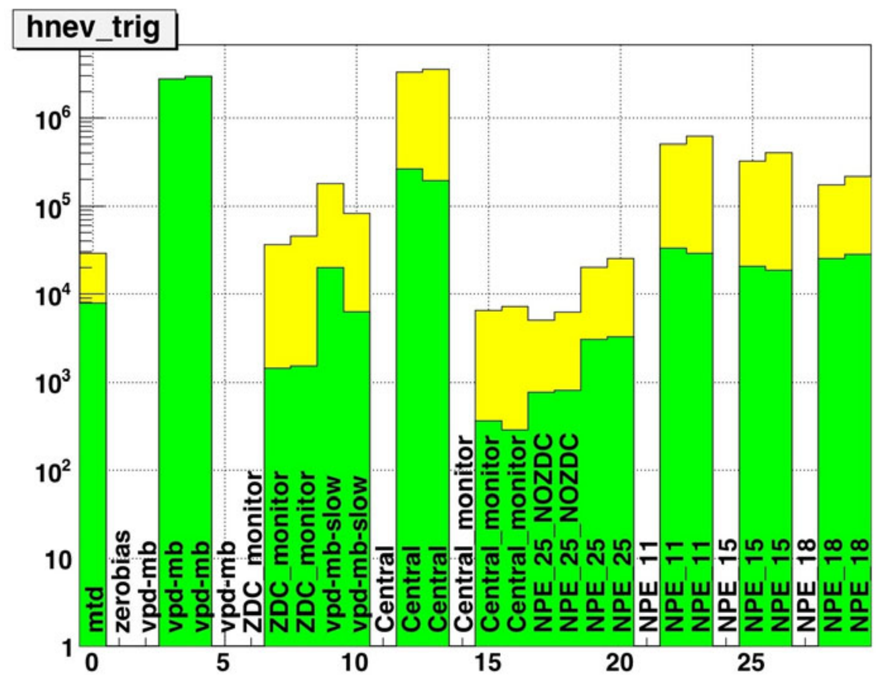
Values from Hiroshi (7.7, 11, 39)
Values from Terry (200)

Integrated within each 10% bin to split into two 5% bins

16 centrality bins
5% wide, 80% to 0%

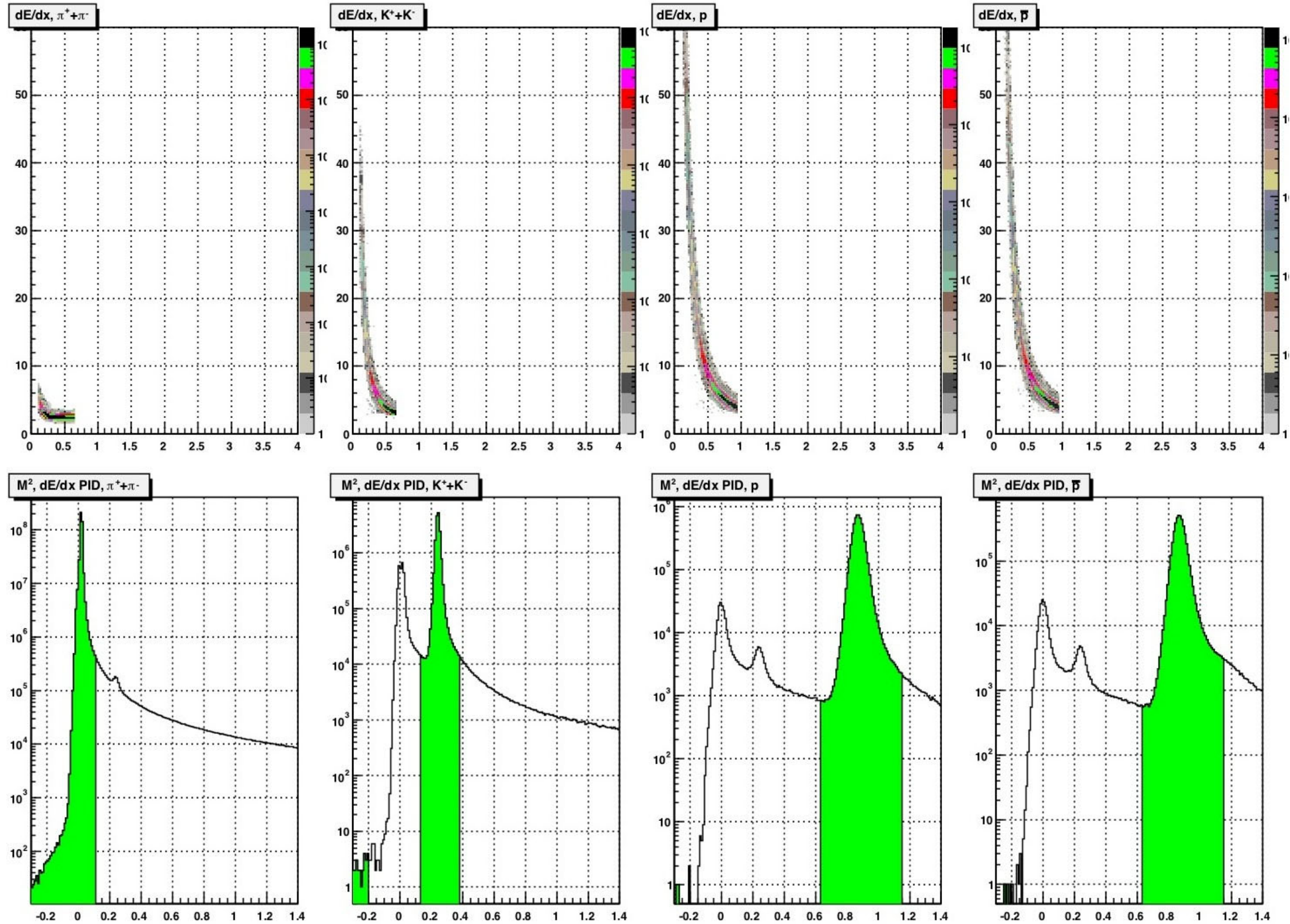
Use only the Minbias triggers from each dataset

200	260001||260011||260021||260031
39	280001
11	310014
7.7	290001||290004



"dE/dx – TOF" PID

2σ cut on π, K, or p with momentum upper limits

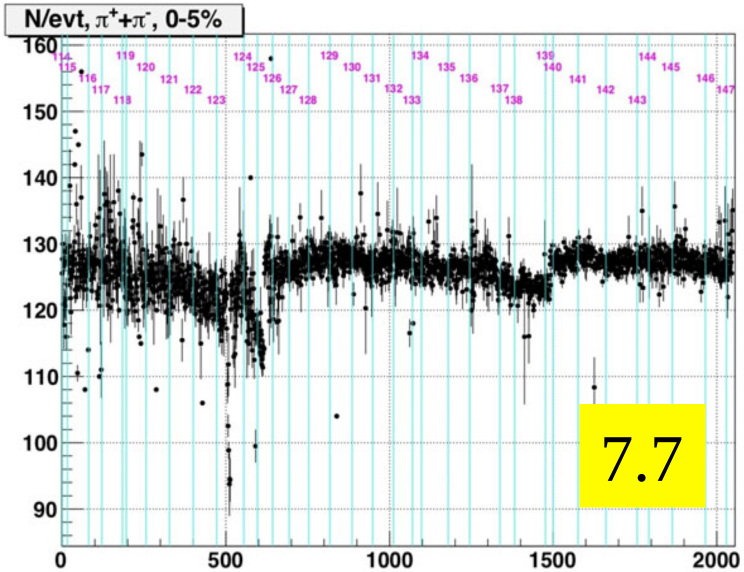


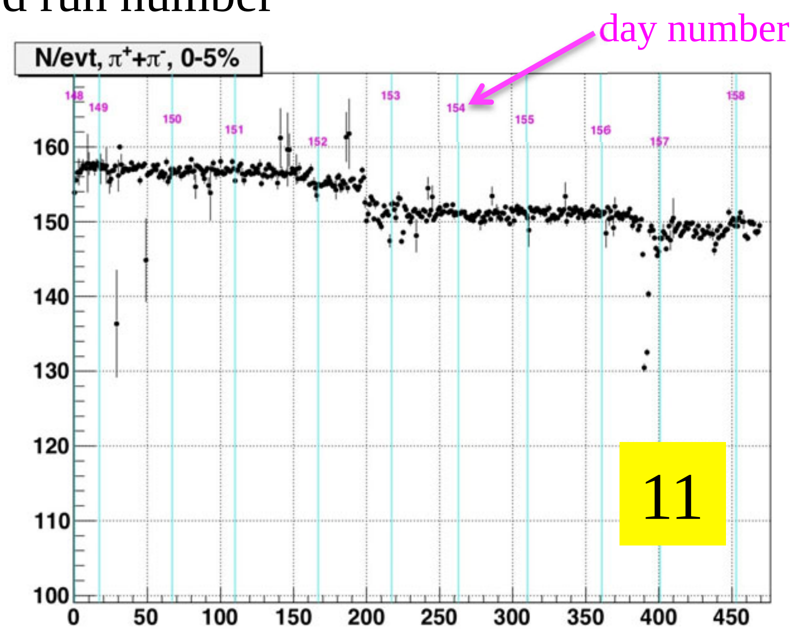
And then look to TOF…. If a TOF value exists, then cut away all but POI…..

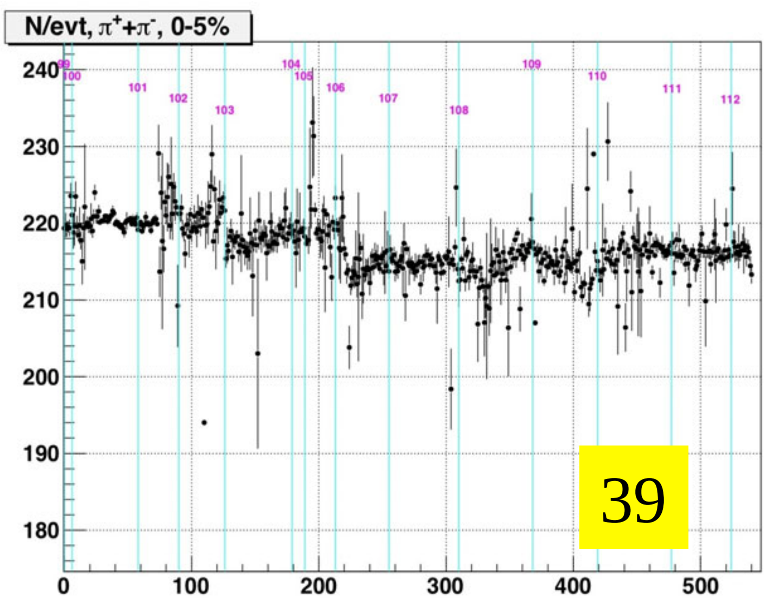
(i.e. PID does not *require* TOF info for a given track, but use it if it exists to clean up the dE/dx PID)

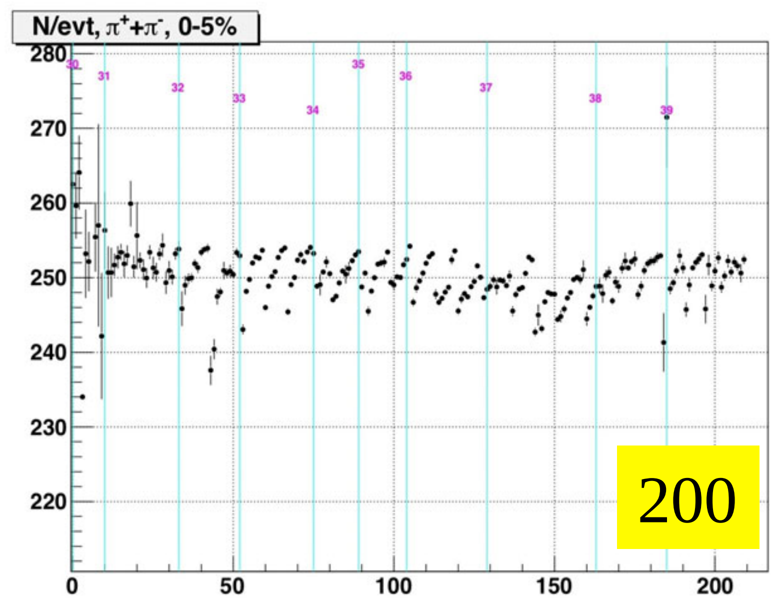
Run Dependence of Particle Rates per Event

$<N_\pi>$ per 0-5% central event versus the ordered run number



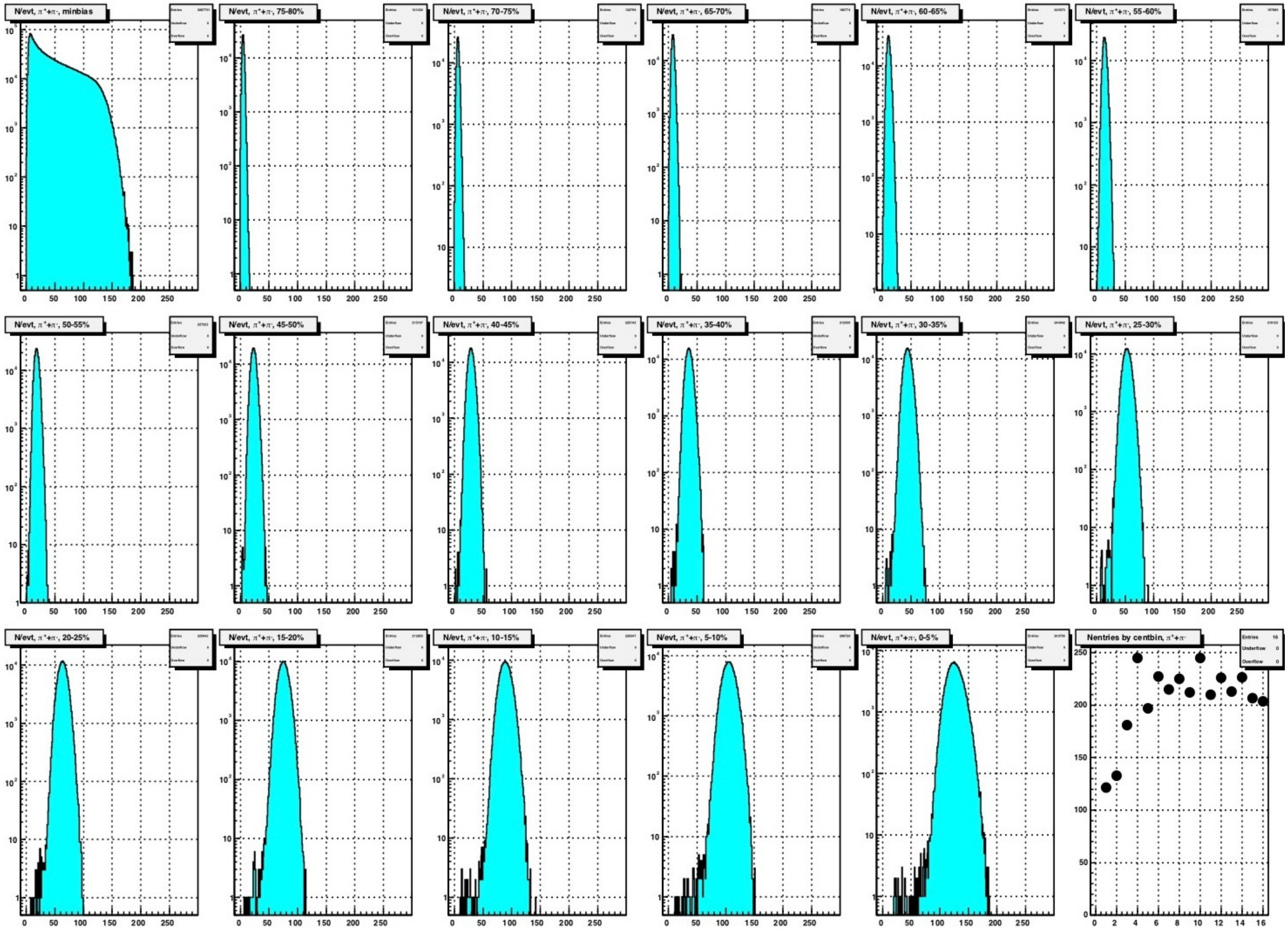






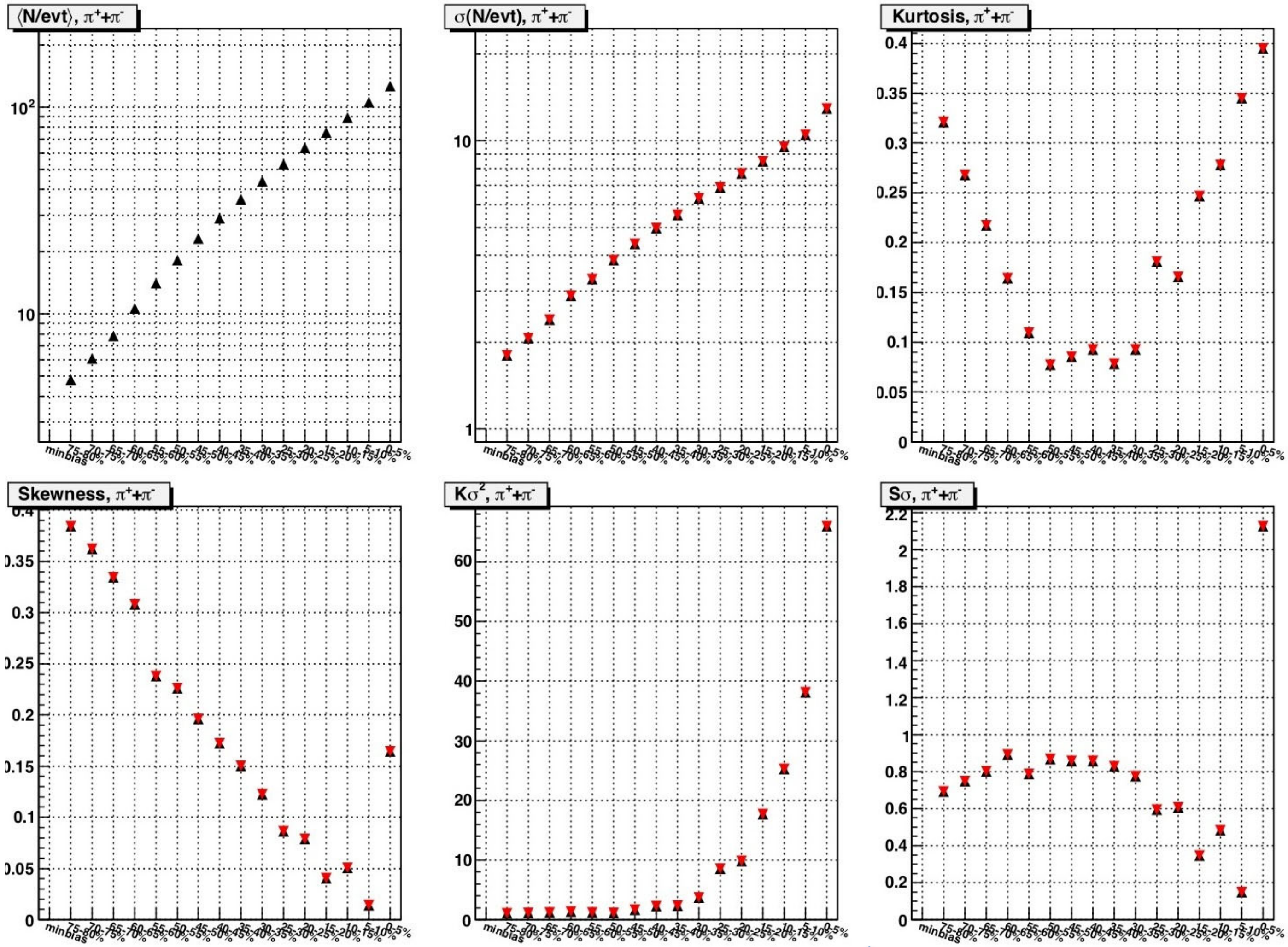
First Pass through the data

Determine mean multiplicities in each centrality bin and save for later use…..



Second Pass through the data….

Read in the mean multiplicities from previous pass and calculate deviates & cumulants
Make sure Mean, RMS, Skewness, and Kurtosis from ROOT & cumulants agree!



ROOT:
h.GetRMS()
h.GetSkewness()
h.GetKurtosis()
(at bin centers….)

Compare to:

RMS:	$\sqrt{\kappa_{2x}}$
Skewness:	$\kappa_{3x}/\kappa_{2x}^{3/2}$
Kurtosis:	$\kappa_{4x}/\kappa_{2x}^{2}$

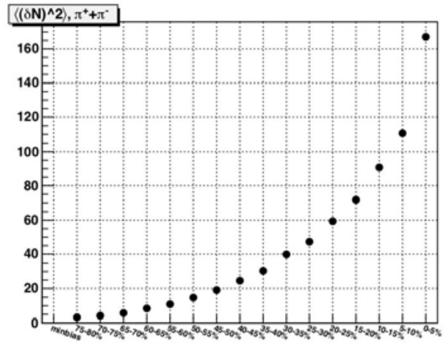
$$S\sigma = \kappa_{3x}/\kappa_{2x}$$
$$K\sigma^2 = \kappa_{4x}/\kappa_{2x}$$

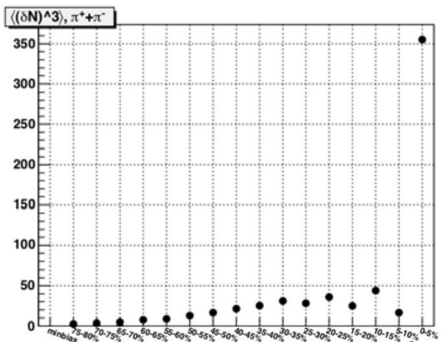
Increasing centrality….

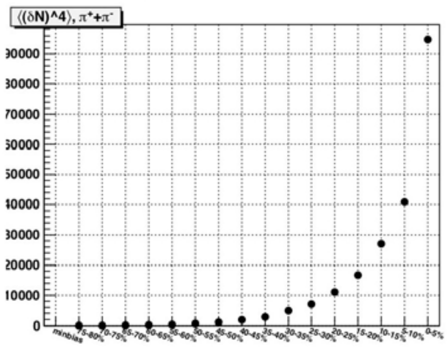
200 GeV data shown…. Make same "test" plots for K, p, pbar & other energies...

Deviate Means

$<\delta x^i>$





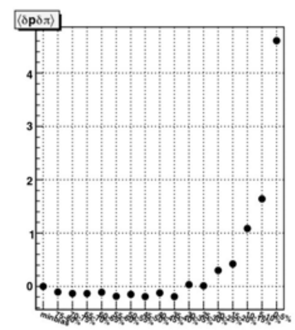
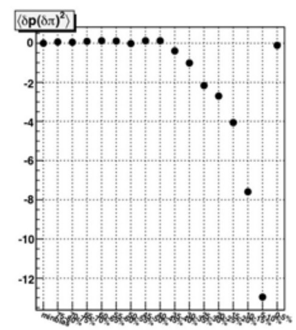
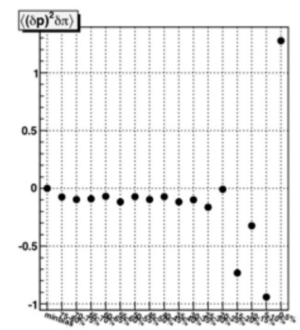
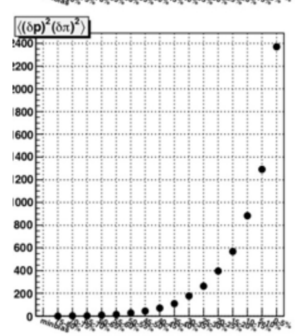
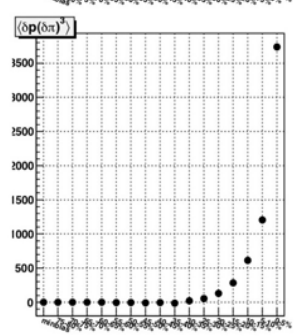


Increasing centrality….

Total pions from 7.7 GeV data shown
Make same plots for
total Kaons
total protons
net protons
at all energies…

Mixed Deviates:
x=p, y=π shown…

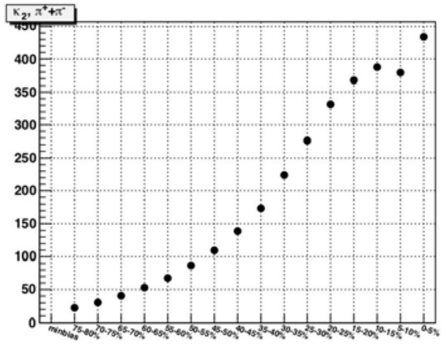
Make same plots for
x=net-protons, y=π

$<\delta x^i \delta y^j>$

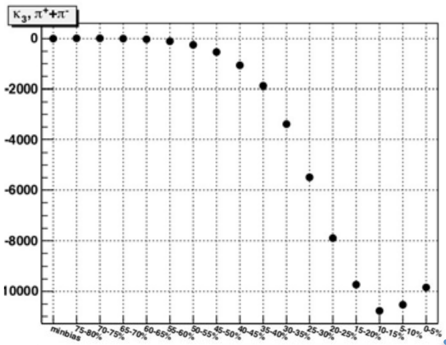
Cumulants and Mixed Cumulants….

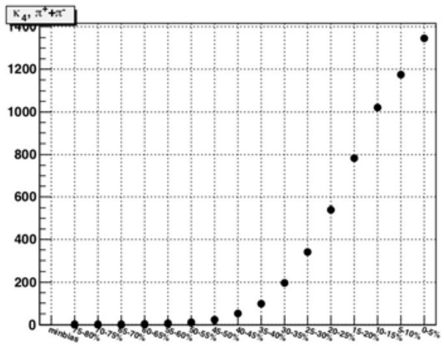
κ_{ix}



Total pions from 7.7 GeV data shown
Make same plots for
- total Kaons
- total protons
- net protons

at all energies…

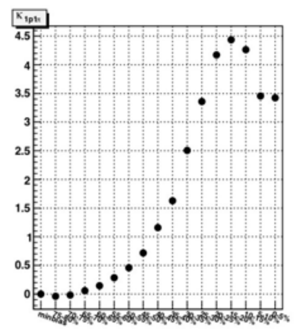


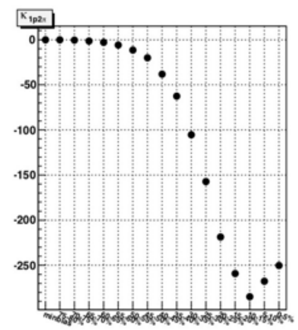


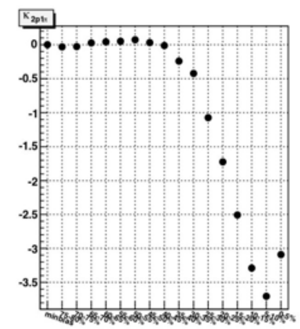
Increasing centrality….

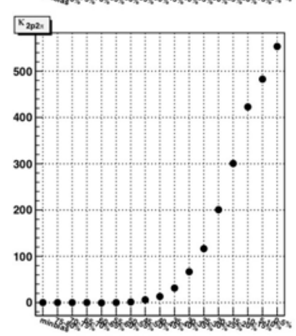
Mixed Cumulants:
x=p, y=π shown…

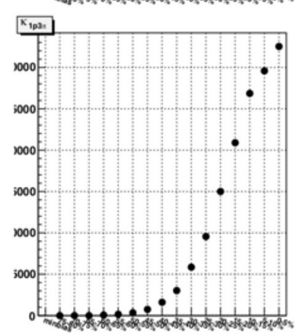
Make same plots for
x=net-protons, y=π







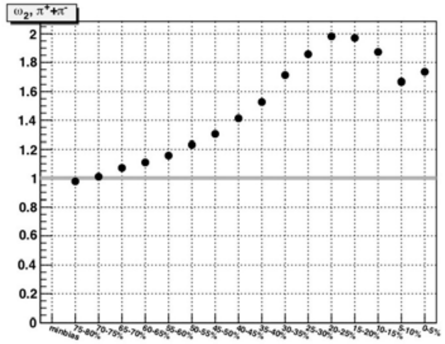


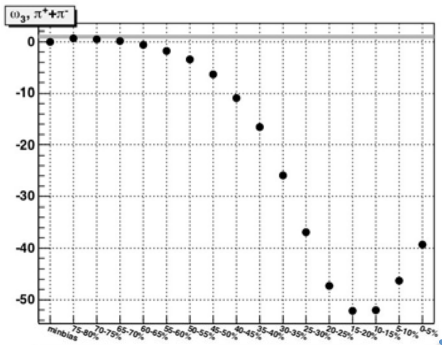


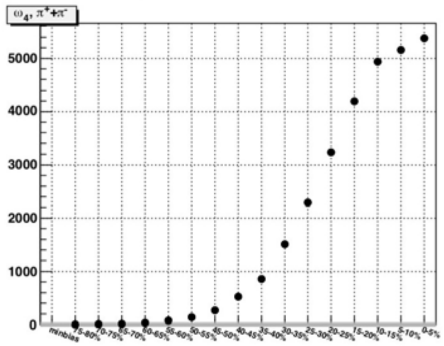
κ_{ixjy}

Normalized Intensive Cumulants….

ω_{ix}







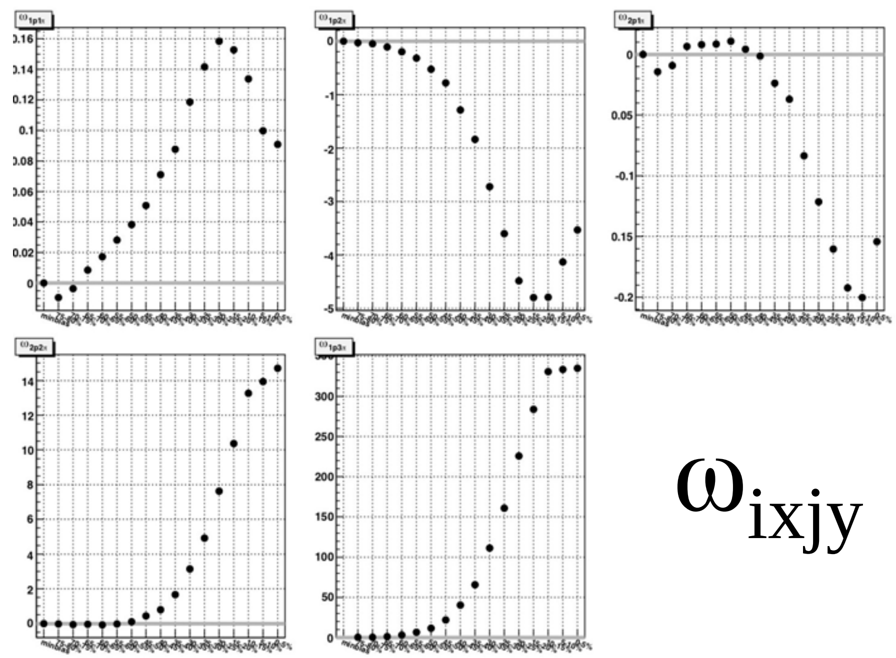
Increasing centrality….

Total pions from 7.7 GeV data shown
Make same plots for
- total Kaons
- total protons
- net protons

at all energies…

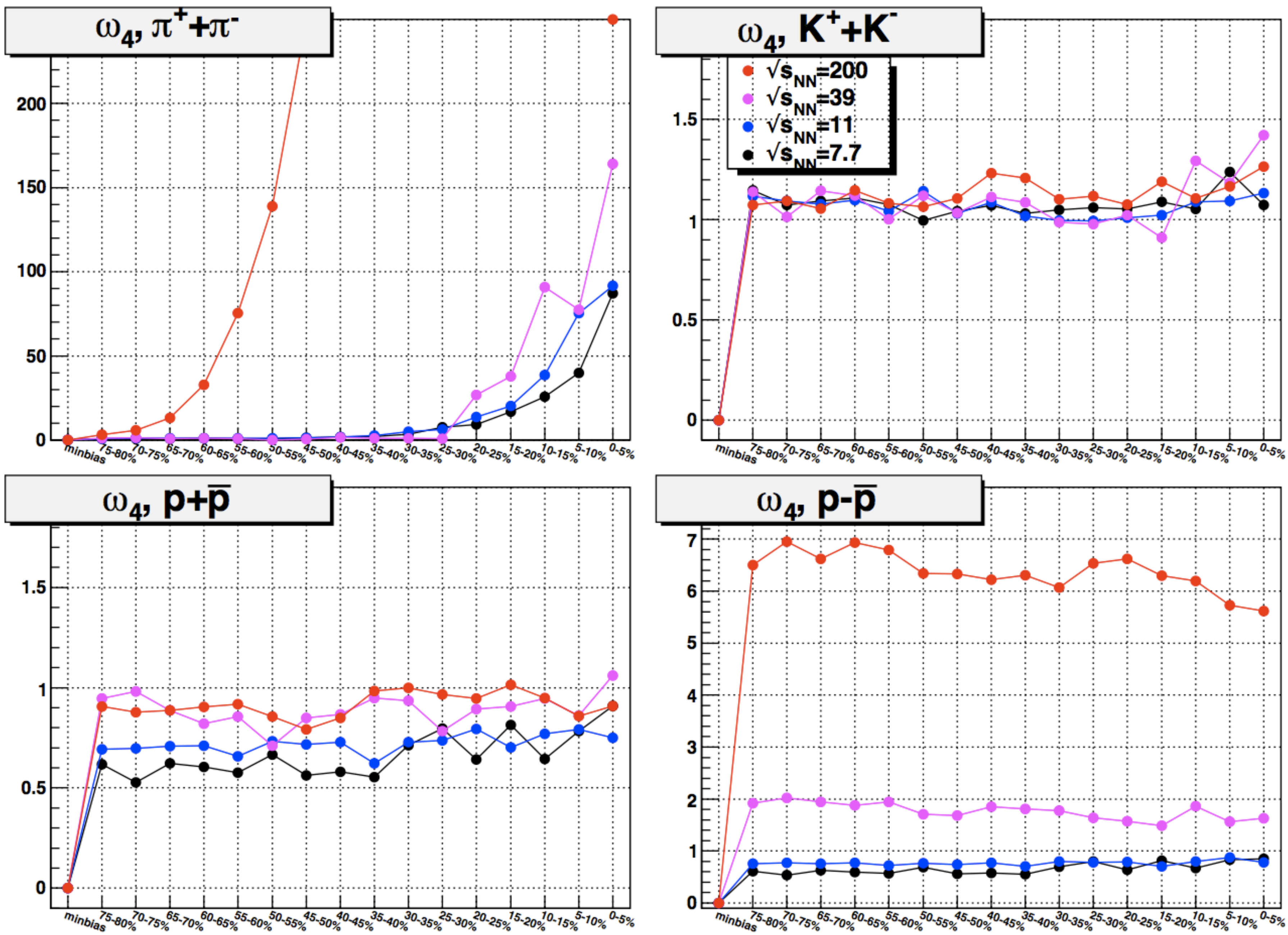
Mixed Norm. Int. Cumulants:
x=p, y=π shown…

Make same plots for
x=net-protons, y=π



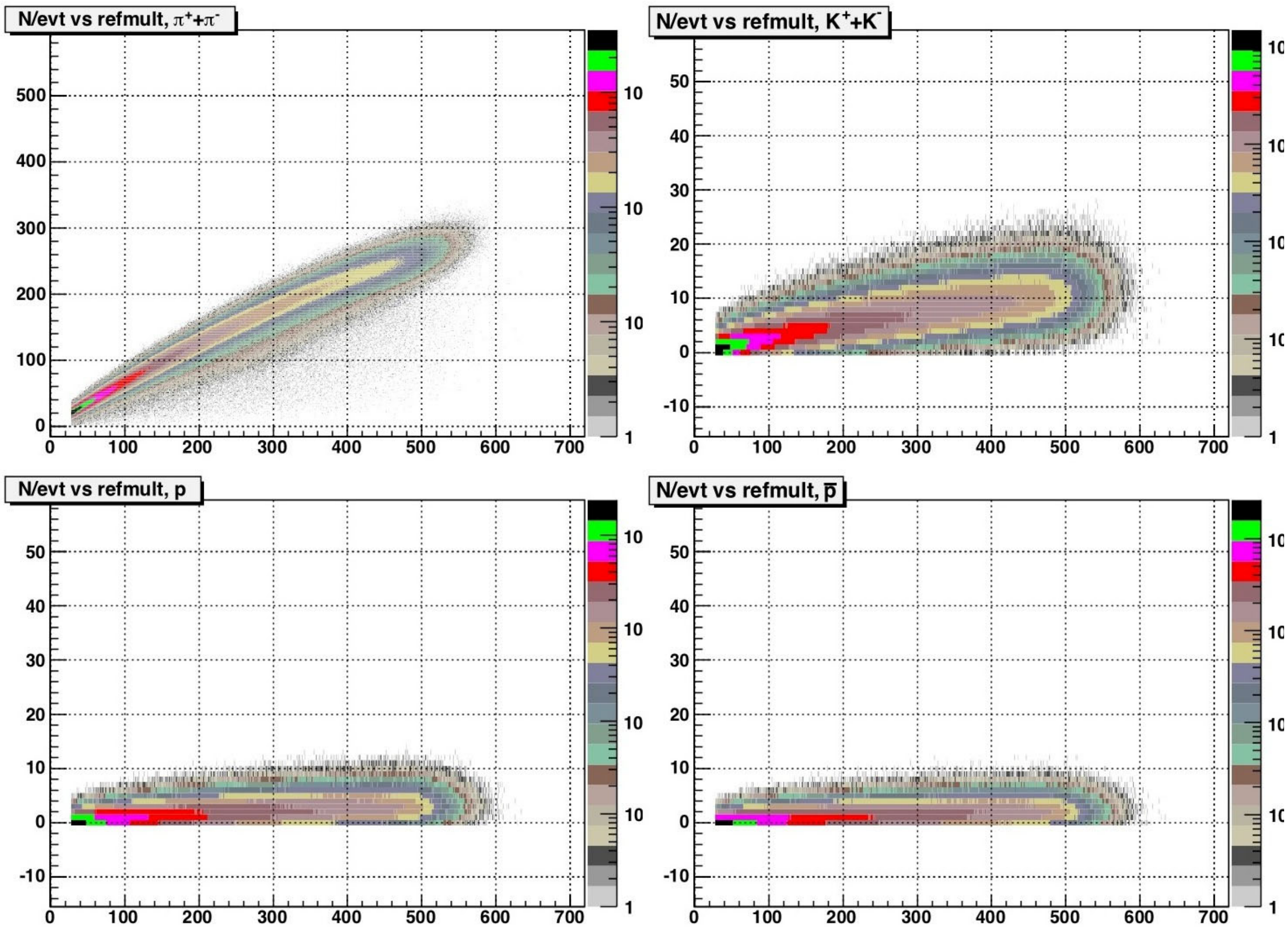
ω_{ixjy}

ω_{4x}


ω4, π++π-
ω4, K++K-
√sNN=200
√sNN=39
√sNN=11
√sNN=7.7
ω4, p+p̄
ω4, p-p̄
minbias
75-80%
70-75%
65-70%
60-65%
55-60%
50-55%
45-50%
40-45%
35-40%
30-35%
25-30%
20-25%
15-20%
10-15%
5-10%
0-5%

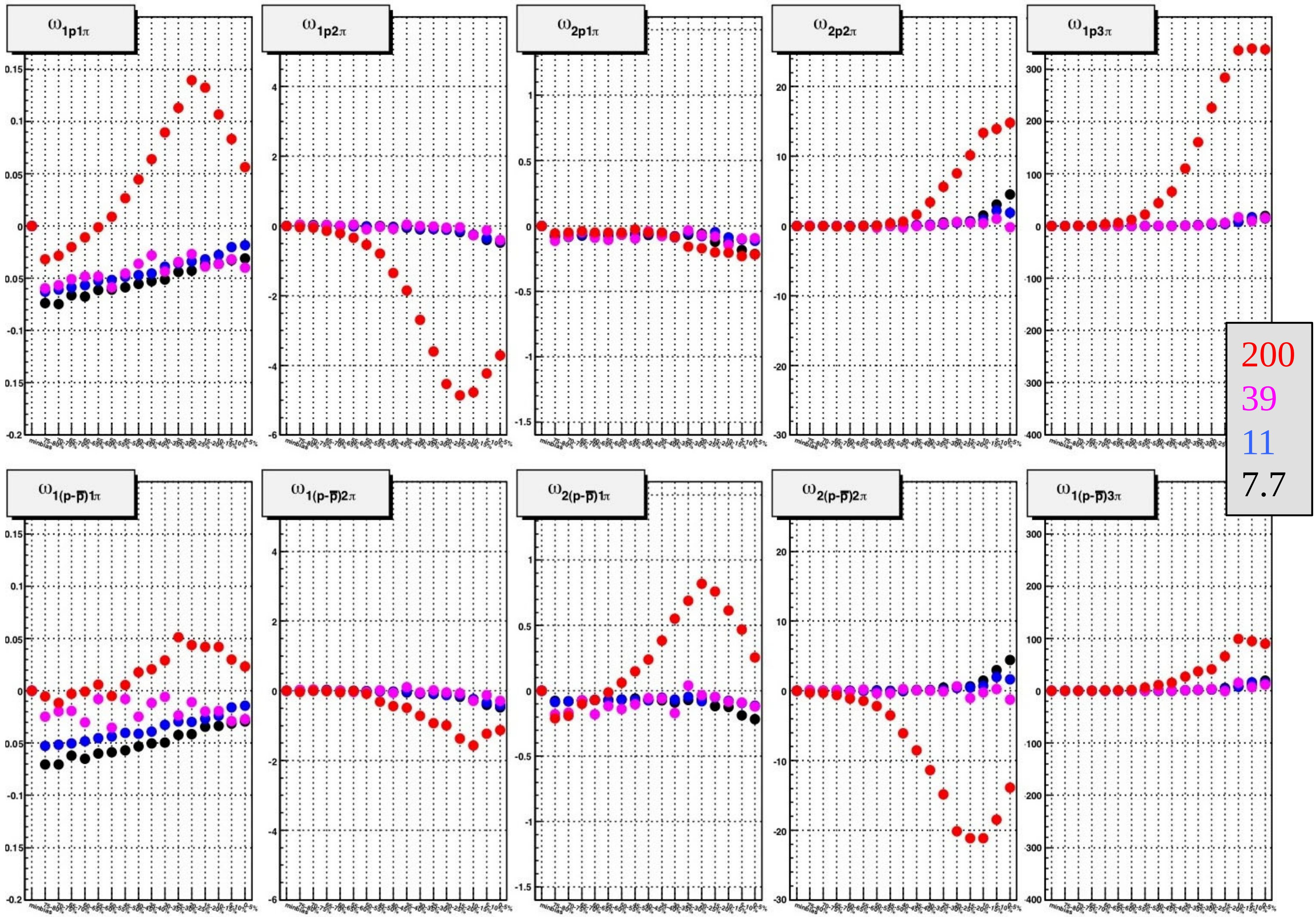
Mixed Normalized Intensive Cumulants from “Randomized” data

At a specific value of refmult, sample from slice to get Nx…
Destroys multiplicity correlations between π, p, & pbar in the event….



Then calculate the same mixed normalized intensive cumulants as before
And plot mixed norm. int. cumulants from “data minus randomized data”….

Mixed Normalized Intensive Cumulants: Data-Random…



To-do List

Lots to do (if there's any interest in this)

- Continue to develop two parallel codes to vet results...
- min P_T cut to avoid proton spallation background...
- dE/dx+TOF to compare to dE/dx−TOF & extend mom'n reach…
- Handle rather obvious run number variations
  - hard cut *vs* doing the whole analysis run-by-run…
- Dependence of cumulants on event and track cuts used...
- uncertainty estimates…
- Better Randomization / Event-mixing for baseline…
- CLT Fitting and model simulations *a la* net-proton PRL…
- Additional cumulants?
  - total $h^{\pm}$, $K^{+}-K^{-}$, total baryons, net baryons, total strangeness,
  - & more mixed cumulants, *i.e.* xy = $K\pi$, Kp, K(p-pbar)

All comments & suggestions are appreciated – *Thanks in advance*...